Th 1. Let $A := I = [a,b] \subseteq \mathbb{R}$ & $f : I \to \mathbb{R}$ cts. Then

(i) f is globally bounded on I :

$$M := \sup\{f(x) : x \in I\} \in \mathbb{R}$$

$$m := \inf\{f(x) : x \in I\} \in \mathbb{R}$$

(ii) $\exists\, x_*, x^* \in I$ s.t.

$$f(x_*) \leq f(x) \leq f(x^*) \;\; \forall x \in I$$

(f attains its max. & min. values)

Proof. Suppose for contradiction that f is not bounded above : $\forall n \in \mathbb{N}\ \exists\, x_n \in I$ s.t. $f(x_n) > n$. Do this, $\forall n$ one has a bounded sequence (x_n) s.t.

$$f(x_n) > n \quad \forall n \in \mathbb{N}.$$

By B-W & order-preserving, $\exists\ x_0 \in [a,b]$ & a convergent subsequence (x_{n_K}) with $\lim_K x_{n_K} = x_0$. Since f is cts at x_0, it follows from the sequential criterion that $\lim f(x_{n_K}) = f(x_0) \in \mathbb{R}$ contradicting $f(x_{n_K}) > n_K \geq K \ \forall K$. Therefore f must be bounded above. Similarly one can show f is also bounded below. (Thus $M, m \in \mathbb{R}$)

(ii). Take $z_n \in I$ s.t. $M - \frac{1}{n} < f(z_n), \forall n \in \mathbb{N}$. Similar as in (i), $\lim_K z_{n_K} = z_0 \in I$ for some subseq. Hence $M - \frac{1}{n_K} < f(z_{n_K}) \to f(z_0)$, and conseq. $M = f(z_0)$, so z_0 has the property required for x^*. Similarly for the remaining part.

Note. Do for bounded closed subset A in place of $[a,b]$.

(Root-Th & Intermediate Value Th)

Th 2. Let $f: I = [a,b] \to \mathbb{R}$ cts. Then

(i) Suppose $f(a)f(b) < 0$. Then $\exists\ c \in (a,b)$ s.t. $f(c) = 0$.

(ii) If $f(a) < k < f(b)$ (or $f(a) > k > f(b)$) then $\exists\ c \in (a,b)$ s.t. $f(c) = k$.

Proof. (1) say $f(a) < 0 < f(b)$. Then (unless already found c with $f(c)=0$) $\exists$ two seq $(a_n), (b_n)$ in I with $a_1 = a$ & $b_1 = b$

$(a_n)\uparrow$, $(b_n)\downarrow$ such that
$$(*)\begin{cases} f(a_n) < 0 < f(b_n) \\ a_n < b_n \\ b_n - a_n = \frac{b-a}{2^{n-1}} \end{cases} \quad \forall n \in \mathbb{N}$$

By the MCT (& order-preserving), with $a_0, b_0 \in I$
$$\lim_n a_n = a_0 \leq b_0 = \lim_n b_n.$$

and it follows from the continuity of f that
$$\lim_n f(a_n) = f(a_0) \ \& \ \lim_n f(b_n) = f(b_0)$$

Noting $a_0 = b_0$ ($\because$ $b_n - a_n \to 0$) it follows from $(*)$ that $f(a_0) = 0$ (by order-preserving property).

To show $(a_n), (b_n)$ do exist (unless already found $c \in (a,b)$ s.t. $f(c)=0$), we sketch below:

At 1st, take $a_1 = a$ & $b_1 = b$. Compute $f\left(\frac{a_1+b_1}{2}\right)$: if $f\left(\frac{a_1+b_1}{2}\right) < 0$ set $a_2 := \frac{a_1+b_1}{2}$ & $b_2 := b_1$

if $f\left(\frac{a_1+b_1}{2}\right) > 0$ set $a_2 := a_1$ & $b_2 = \frac{a_1+b_1}{2}$

Having $f(a_2) < 0 < f(b_2)$ with $b_2 - a_2 = \frac{b_1 - a_1}{2}$, set
$$\left.\begin{aligned} a_3 &:= \frac{a_2+b_2}{2} \\ b_3 &= b_2 \end{aligned}\right\} \text{ if } f\left(\frac{a_2+b_2}{2}\right) < 0$$

Inductively continue.

(ii). Consider (the cts function) $f(\cdot) - k$ & notice that

$$\left(f(\cdot)-k\right)(a) < 0 \text{ \&}$$

$$\left(f(\cdot)-k\right)(b) > 0$$

so apply (i') to obtain $c \in (a,b)$ s.t.

$$\left(f(\cdot)-k\right)(c) = 0$$

i.e. $f(c) = k$.

Cor. (i) cts image of a bounded closed ~~interval~~ interval is a bounded closed interval. Indeed

$$f\left([a,b]\right) = [m, M] = [f(x_*), f(x^*)]$$

in the notation of Th 1

(Hint: Th 2 is needed).

(ii) cts image of an interval is an interval: let $f: J \to \mathbb{R}$ be cts & J be an interval. Then

$$f(J) := \{f(x) : x \in J\} \text{ is an interval}$$

because it is "order-convex" by Th 2.